Optical Cavity Designs for Interferometric Gravitational Wave Detectors

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Assignments

1. Assuming a cavity of 4km with an ITM of 1934m radius of curvature and an ETM of 2245m radius of curvature. Calculate the following parameters:
   - Waist size and position
   - Spot size on both ETM and ITM
   - Stability g factor for both mirrors and the whole cavity
   - Free spectral range
   - High order mode spacing

2. To the previous cavity we add two mirrors with the following radius of curvature: 34m and -4.41m. Distance between ITM and the first mirror is 29.27m, from this mirror to the second mirror is 15.21m.

   Assuming a vacuum environment and fused silica substrate determine the radius of curvature of the wavefront at a distance of 15.8m from the second mirror.
Gaussian Beam

Complex beam parameter:

\[ \frac{1}{q(z)} = \frac{1}{R(z)} - j \frac{\lambda}{\pi \omega^2(z)} \]

Spot size:

\[ \omega^2(z) = \omega_o^2 \left[ 1 + \left( \frac{\lambda z}{\pi \omega_o^2} \right)^2 \right] \]

Rayleigh range:

\[ z_R = \frac{\pi \omega_o^2}{\lambda} \]

Beam divergence:

\[ \theta \approx \frac{\lambda}{\pi \omega_o} \]

Radius of curvature of the wavefront:

\[ R(z) = z \left[ 1 + \left( \frac{\pi \omega_o^2}{\lambda z} \right)^2 \right] \]

Gouy phase:

\[ \psi(z) = \arctan \left( \frac{z}{z_R} \right) \]
Transverse Modes

Higher Order Modes

- Hermite – Gaussian:

\[
U_{m,n}(x, y, z) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \sqrt{\frac{\exp\{j(2m+2n+1)\psi(z)\}}{2^m 2^n m!n! \omega(z)^2}} \times H_m\left(\frac{\sqrt{2x}}{\omega(z)}\right) H_n\left(\frac{\sqrt{2y}}{\omega(z)}\right) \exp\left\{-j2kz - jk\left(\frac{x^2 + y^2}{2R(z)}\right) - \frac{x^2 + y^2}{\omega(z)^2}\right\}
\]

- Laguerre – Gaussian:

\[
U_{l,m}(r, \phi, z) = \sqrt{\frac{4l!}{(1 + \delta_{0,m})\pi(l + m)!}} \left(\frac{\exp\{j(2l+m+1)\psi(z)\}}{\omega(z)}\right) \cos(m\phi) \times \left(\frac{\sqrt{2r}}{\omega(z)}\right)^m L_{l,m}\left(\frac{2r^2}{\omega(z)^2}\right) \exp\left\{-jkr - jk\left(\frac{r^2}{2R(z)}\right) - \frac{r^2}{\omega(z)^2}\right\}
\]
Transverse Modes

Higher Order Modes

- Hermite – Gaussian:

\[ U_{m,n}(x, y, z) = \left( \frac{1}{2\pi} \right)^{\frac{1}{2}} \sqrt{\frac{\exp\{j(2m + 2n + 1)\psi(z)\}}{2^m 2^n m! n! \omega(z)^2}} \times H_m \left( \frac{\sqrt{2x}}{\omega(z)} \right) \]

- Laguerre – Gaussian:

\[ U_{l,m}(r, \phi, z) = \sqrt{\frac{4l!}{(1 + \delta_{0,m})\pi(l + m)!}} \left( \frac{\exp\{j(2l + m + 1)\psi(z)\}}{\omega(z)} \right) \cos(m\phi) \]
Frequency of Transverse Modes

Phase shift:

total phase shift from one end of the cavity to the other, including the Gouy phase

\[ \phi(z_1 - z_2) = kL - (m + n + 1) \times [\psi(z_1) - \psi(z_2)] \]

\[ \psi(z_2) - \psi(z_1) = \arccos(\pm \sqrt{\frac{g_1}{g_2}}) \]

Beam radius and Gouy phase shift along the propagation direction for a beam in air with 1064 nm wavelength and 100\(\mu\)m radius at the waist. The positions at plus and minus the Rayleigh length are marked.
Frequency of Transverse Modes

Frequency of higher order modes for a given axial mode:

\[ \nu_{qmn} = q \nu_0 + \frac{\nu_0}{\pi} (m + n + 1) \arccos \left( \pm \sqrt{g_1 g_2} \right) \quad \nu_0 = \text{free spectral range} \]

This frequency and the finesse of the cavity defines the suppression factor

\[ S_{mn} = \left[ 1 + \frac{4 \mathcal{F}^2}{\pi^2} \sin^2 \left( \frac{\Delta \nu_{mn}}{\nu_0} \right) \right]^{\frac{1}{2}} \]

Therefore we can define a transmission factor

\[ T_{mn} = \frac{t_1 t_2}{(1 - r_1 r_2)} \frac{1}{\left[ 1 + \frac{4 \mathcal{F}^2}{\pi^2} \sin^2 \left( \frac{\Delta \nu_{mn}}{\nu_0} \right) \right]^{\frac{1}{2}}} \]
Higher Order Modes Tuning

Assuming symmetric cavity

\[ R = \infty \quad \text{and} \quad g_{\text{mirror}} = 1 \]

\[ L = 4000 \text{ m} \quad \text{and} \quad \Psi = 0 \text{ rad} \]

\[ \Delta f_{\text{TEM}} = 0 \text{ kHz} \]

\[ \Delta f_{\text{ax}} \]

Near Planar!
Higher Order Modes Tuning

Assuming symmetric cavity

\[ R = 54416 \text{ m} \quad g_{\text{mirror}} = 0.926 \]

\[ L = 4000 \text{ m} \quad \Psi = 0.39 \text{ rad} \]

\[ \Delta f = 0 \text{ kHz} \quad \Delta f_{\text{TEM}} = 4.6 \text{ kHz} \]
Higher Order Modes Tuning

Assuming symmetric cavity

\[ R = 4000 \text{ m} \]
\[ L = 4000 \text{ m} \]
\[ g_{\text{mirror}} = 0 \]
\[ \Psi = \frac{\pi}{2} \text{ rad} \]

\[ \Delta f_{\text{ax}} \]
\[ \Delta f_{\text{TEM}} = 18.7 \text{ kHz} \]

Con-focal!
Higher Order Modes Tuning

Assuming symmetric cavity

\[ R = 2076.5 \, \text{m} \quad \text{and} \quad \Delta f_{\text{TEM}} = 32.9 \, \text{kHz} \]

\[ \Psi = 2.76 \, \text{rad} \]

\[ \Delta f_{\text{ax}} = 4.6 \, \text{kHz} \]

\[ \Delta f_{\text{TEM}} = 32.9 \, \text{kHz} \]

\[ f \, [\text{kHz}] \]

\[ 0 \quad 37.47 \quad 74.95 \quad f \, [\text{kHz}] \]
**Higher Order Modes Tuning**

Assuming symmetric cavity

\[ R = 2000 \text{ m} \]
\[ L = 4000 \text{ m} \]
\[ g_{\text{mirror}} = -1 \]
\[ \Psi = \pi \text{ rad} \]

\[ \Delta f_{\text{TEM}} = 37.5 \text{ kHz} \]

Concentric!
Higher Order Modes Tuning

\[ g = \cos^2(\Psi) \]
Ray Transfer Matrices

Matrix representation of ray trace:

\[
\begin{pmatrix}
  x_2 \\
  x_2'
\end{pmatrix} =
\begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_1'
\end{pmatrix}
\]

\(x_1\) corresponds to the position and \(x_1'\) corresponds to the slope (or angle) of the beam.

It is necessary to create one matrix per optical element.

Ray transfer matrices for Gaussian beams

\[
\frac{1}{q} = \frac{1}{R} - j \frac{\lambda}{\pi \omega^2}
\]

\[
\begin{pmatrix}
  q_2 \\
  1
\end{pmatrix} = k
\begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix}
\begin{pmatrix}
  q_1 \\
  1
\end{pmatrix}
\implies
\frac{1}{q_2} = \frac{C + D}{A + B} \frac{1}{q_1}
\]

# Ray Transfer Matrices

For simple optical components

<table>
<thead>
<tr>
<th>Element</th>
<th>Matrix</th>
<th>Remarks</th>
</tr>
</thead>
</table>
| Propagation in free space or in a medium of constant refractive index | \[
\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}
\] | \(d = \text{distance}\)                                                  |
| Refraction at a flat interface               | \[
\begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix}
\] | \(n_1 = \text{initial refractive index}\) \(n_2 = \text{final refractive index}\). |
| Refraction at a curved interface             | \[
\begin{pmatrix} 1 & 0 \\ \frac{n_1-n_2}{Rn_2} & \frac{n_1}{n_2} \end{pmatrix}
\] | \(R = \text{radius of curvature}, R > 0 \text{ for convex (centre of curvature after interface)}\) \(n_1 = \text{initial refractive index}\) \(n_2 = \text{final refractive index}\). |
| Reflection from a flat mirror                | \[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\] | Identity matrix                                                        |
| Reflection from a curved mirror              | \[
\begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix}
\] | \(R = \text{radius of curvature}, R > 0 \text{ for concave}\)          |
| Thin lens                                    | \[
\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}
\] | \(f = \text{focal length of lens where } f > 0 \text{ for convex/positive (converging) lens. Valid if and only if the focal length is much greater than the thickness of the lens.}\) |
In 1897 Fabry and Perot constructed an optical resonator for use as an interferometer.
Fabry-Perot Cavity

Original (1898) model of the Fabry-Perot interferometer.
Fabry-Perot Cavity

Stable Resonator

Free Spectral Range: Finesse:

\[ FSR = \frac{c}{2L} = \frac{3 \times 10^8}{2 \left(4 \times 10^3\right)} = 37.5 \text{ kHz} \]

\[ \mathcal{F} = \frac{\Delta \omega_{ax}}{\Delta \omega_{cav}} = \pi \frac{\sqrt{r_1 r_2}}{(1 - r_1 r_2)} \]

Refer to notes from lecture 6 for complete analysis
Fabry-Perot Cavity

Stability factor:
\[ g_1 = 1 - \frac{L}{R_1} \quad g_2 = 1 - \frac{L}{R_2} \]

Cavity stability:
\[ g_1 g_2 = \left(1 - \frac{L}{R_1}\right)\left(1 - \frac{L}{R_2}\right) \]

Stable cavity:
\[ 0 \leq g_1 g_2 \leq 1 \]

Gouy phase:
\[ g = \cos^2(\psi) \]
Advanced Interferometers

Laser → M1 → M2 → PRM → BS → ITM_Y → ETM_Y → ITM_X → ETM_X → To Detector Bench

M3 → Laser

SRM
Advanced Interferometers

Input Mode-cleaner

Power Recycling Cavity

Signal Recycling Cavity

Output Mode-cleaner

To Detector Bench
Length Degrees of Freedom

Degrees of freedom

\[ L_+ = (L_x + L_y)/2 \]
\[ L_- = (L_x - L_y)/2 \]
\[ l_+ = l_{pr} + (l_x + l_y)/2 \]
\[ l_- = (l_x - l_y)/2 \]
\[ l_{src} = l_{sr} + (l_x + l_y)/2 \]
**Length Sensing and Control**

**Carrier:**
Resonant in PRC and Arms

**RF Sideband f1:**
Resonant in PRC

**RF Sideband f2:**
Resonant in PRC and SRC
Rule of Thumb

• Carrier should be resonant in the arms and the PRC.

• Carrier resonant in the SRC for resonant sideband extraction (RSE), and anti-resonant for signal recycling.

• SB1 should be nearly anti-resonant in the arms, and resonant in the PRC.

• SB2 also nearly anti-resonant in the arms, and resonant in the PRC.

• One of the SB should be resonant in the SRC and the other nearly anti-resonant.
Sidebands Selection

For transmission of modulation sidebands by the mode-cleaner, $L_{IMC}$ and $f_m$ must satisfy:

$$f_m = n_1 \frac{c}{2L_{IMC}}$$

For sideband coupling into the recycling cavity, $L_{PRC}$ and $f_m$ must satisfy:

$$f_m = \left(n_2 + \frac{1}{2}\right) \frac{c}{2L_{PRC}}$$

Sideband must not resonate inside the main arms, but also not exactly anti-resonant:

$$n_3 = f_m \left(\frac{2L_{Arm}}{c}\right)$$

$n_3$ not an integer
Signal recycling cavity is different and depends on the operation scheme selected for the interferometer.

Only the higher frequency $f_{m2}$ is used to determine the length of the SRC.

$$L_{SRC} + \Delta L_{SRC} = \frac{c}{2\pi f_{m2}}(n_4 \pi + \phi_s)$$

$$\phi_s = 0 \implies \text{Resonant Sideband Extraction}$$

In order to do this we also need to introduce a difference in the arms lengths in the recycling cavities, known as Schnupp Asymmetry.

$$l \equiv |l_x - l_y| \implies l = \frac{c}{4f_{m2}}$$
A ring cavity will act as an optical filter transmitting only the fundamental mode $\text{TEM}_{00}$.

A field distribution symmetric with respect to the vertical axis closes in itself whenever the total cavity length ($L$) is an integer multiple ($q$) of the optical wavelength ($\lambda$).

Symmetry with respect to the $y$-axis implies that the spatial dependences of the field with $x$ is an even function

$$E(x,y) = -E(x,y)$$

If the field distribution is anti-symmetric with respect to the vertical $y$-axis, the resonance condition is achieved when $L = (q+1/2)\lambda$ for a cavity formed by an odd number of mirrors and when $L = q\lambda$ for a cavity with an even number of mirrors.

$$E(x,y) = -E(-x,y)$$
System of coordinate:

- $z$ is along the direction of propagation
- $x$ is parallel to the plane of incidence on the cavity mirrors
- $y$ is perpendicular to the same plane of incidence
Frequency of Higher Order Modes

We can formalise this effect by writing two equations:

\[ 2kL - 2(m + n + 1)\arccos\left(\sqrt{g}\right) = 2\pi q \quad \Leftrightarrow \ m \ \text{is even} \]

\[ 2kL - 2(m + n + 1)\arccos\left(\sqrt{g}\right) = 2\pi \left( q + \frac{1}{2} \right) \quad \Leftrightarrow \ m \ \text{is odd} \]

Or a more general expression:

\[ 2kL - 2(m + n + 1)\arccos\left(\sqrt{g}\right) - \pi \left( 1 - (-1)^m \right) = 2\pi q \]

\[ \frac{(1 - (-1)^m)}{2} = 0 \quad \text{for } m = 0, 2, 4, 6... \]

\[ \frac{(1 - (-1)^m)}{2} = 1 \quad \text{for } m = 1, 3, 5... \]
Frequency of Higher Order Modes

From here we can deduce the frequency of any higher order mode

\[ 2kL - 2(m + n + 1) \arccos(\sqrt{g}) - \pi \frac{(1 - (-1)^m)}{2} = 2\pi q \]

\[ k = \frac{\omega}{c} \]

\[ FSR = v_0 = \frac{c}{2L} \]

\[ v_{qmn} = q v_0 + \frac{v_0}{\pi} (m + n + 1) \arccos(\sqrt{g}) + \frac{v_0}{2} \frac{(1 - (-1)^m)}{2} \]
The delta frequency between the fundamental mode and any higher order mode is then given by:

\[ m = n = 0 \Rightarrow \nu_{00} = q \nu_0 + \frac{\nu_0}{\pi} \arccos(\sqrt{g}) \]

\[ \Delta \nu_{mn} = \nu_{mn} - \nu_{00} = q \nu_0 + \frac{\nu_0}{\pi} (m + n + 1) \arccos(\sqrt{g}) + \frac{\nu_0}{2} \frac{(1 - (-1)^m)}{2} - q \nu_0 - \frac{\nu_0}{\pi} \arccos(\sqrt{g}) \]

\[ \Delta \nu_{mn} = \left( \frac{\nu_0}{\pi} + \frac{\nu_0}{\pi} (m + n) - \frac{\nu_0}{\pi} \right) \arccos(\sqrt{g}) + \frac{\nu_0}{2} \frac{(1 - (-1)^m)}{2} \]

\[ \Delta \nu_{mn} = \frac{\nu_0}{\pi} (m + n) \arccos(\sqrt{g}) + \frac{\nu_0}{2} \frac{(1 - (-1)^m)}{2} \]
Suppression of Higher Order Modes

How effective the cavity will be for filtering the higher order modes is then given by the suppression factor of the cavity

\[
S_{mn} = \left[ 1 + \frac{4 \mathcal{F}^2}{\pi^2} \sin^2 \left( \frac{2 \pi \Delta v_{mn}}{c} L \right) \right]^{\frac{1}{2}}
\]

\[
S_{mn} = \left[ 1 + \frac{4 \mathcal{F}^2}{\pi^2} \sin^2 \left( \pi \frac{\Delta v_{mn}}{v_0} \right) \right]^{\frac{1}{2}}
\]

\[
S_{mn} = \left[ 1 + \frac{4 \mathcal{F}^2}{\pi^2} \sin^2 \left( (m+n) \arccos(\sqrt{g}) + \frac{\pi}{2} \frac{(1-(-1)^m)}{2} \right) \right]^{\frac{1}{2}}
\]
Transmission of Higher Order Modes

The suppression factor also determines the transmission factor for the cavity.

The transmission of the fundamental mode by this factor will determine the transmission of any given higher order mode.

\[
T_{mn} = T_{00} \frac{1}{S_{mn}}
\]

\[
T_{00} = \frac{t_1 t_2}{(1 - r_1 r_2 r_3)}
\]

\[
T_{mn} = \frac{t_1 t_2}{(1 - r_1 r_2 r_3)} \left[ 1 + \frac{4F^2}{\pi^2} \sin^2 \left( (m + n) \arccos \left( \sqrt{1 - \frac{L}{R}} \right) \pm \frac{\pi}{2} \frac{1 - (-1)^m}{2} \right) \right]^{\frac{1}{2}}
\]
Frequency of Higher Order Modes

High Order Modes Frequency

Transmission factor

Frequency of Higher Order Modes

TEM00\(_{(1)}\)

TEM00\(_{(2)}\)

TEM01

TEM02

TEM20

TEM21

TEM03

TEM10

TEM40

TEM22

TEM04

TEM11

TEM12

TEM30

TEM31

TEM13

TEM14

TEM23

TEM32

TEM41

TEM42

TEM43

TEM44

Frequency [Hz]
Mode-cleaner Geometry

- Concave end mirror
- Incident angle 0.579°

Flat Mirrors used as input and output couplers

Incident angle 44.712°
Coating and Substrate Absorptions

High reflectivity coating absorption produces astigmatic thermal lensing. The spot ellipticity produce different distribution between X and Y axis.

M2 used as output coupler the diagonally transmitted beam produces strong astigmatic thermal lensing.
Eccentricity Variation with Power

M1/M2 = Flat
M3 = 22.5 m

M1/M2 = 1000 m
M3 = 23.3 m

M1/M2 = 470 m
M3 = 23.3 m
Output Mode-cleaner

- Some light will leak to the dark port reducing fringe contrast and increasing the noise.

- An OMC should reject all the components of the contrast defect.

- The OMC design will depend on the readout configuration Heterodyne (RF) or Homodyne (DC).

- RF uses the sidebands as local oscillator → long OMC with suspended mirrors.

- DC uses a small amount of carrier light as local oscillator → no need of sidebands → small OMC in a bow-tie configuration.
Output Mode-cleaner
Example: GEO 600 design

- 4 mirrors cavity
- Finesse of 150
- Cavity g factor of 0.735
- Round trip length of 66 cm

Results from Finesse:
cavity is stable! Eigenvalues:
qx=(-0.0350013 + 0.581682i), w0x=443.85258um zx=-35.001292mm
qy=(-0.0350013 + 0.586335i), w0y=445.6241um zy=-35.001317mm
finesse : 155.114, round-trip power loss: 0.03969604
opt. length: 658.181299mm, FSR: 455.48614MHz
FWHM: 2.9364545MHz (pole: 1.4682273MHz)

Preliminary results of transmission:
- Sidebands: 0.0096
  - TEM10 : 0.0039
  - TEM20 : 0.00018
  - TEM30 : 0.00010
  - TEM40 : 0.0016
  - TEM50 : 0.0060
## Arm Cavity Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cavity Length</td>
<td>4000 m</td>
</tr>
<tr>
<td>ITM (ETM) radius of curvature</td>
<td>2076 m</td>
</tr>
<tr>
<td>ITM (ETM) diameter</td>
<td>32 cm</td>
</tr>
<tr>
<td>Cavity $g$-factor</td>
<td>0.859</td>
</tr>
<tr>
<td>Waist size radius</td>
<td>11.491 mm</td>
</tr>
<tr>
<td>Spot size radius</td>
<td>60.057 mm</td>
</tr>
<tr>
<td>Free spectral range</td>
<td>37.47 kHz</td>
</tr>
<tr>
<td>High order modes separation</td>
<td>32.88 kHz</td>
</tr>
</tbody>
</table>
Power Recycling Cavity

Case 1: Marginally stable

Original Advanced LIGO design: R1 = R2 = 2076m with arm length of 4000m

Waist radius = 11.491 mm
Mirror spot size radius = 60.057 mm
Waist position = 2000 m
FSR = 37.5 kHz
HOM frequency gap = 32.9 kHz
Rayleigh range = 389.872 m
Beam radius = 2076 m

\[ q(z) = z + jz_R = 2000 + j389.872 \]